Dvojno nihalo

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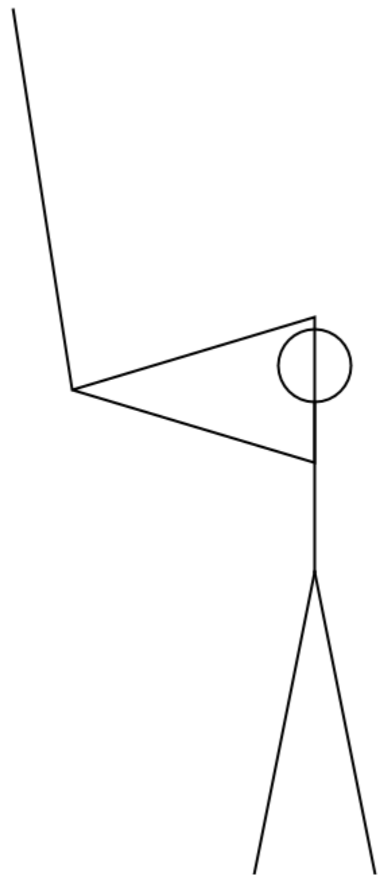
Stalno strokovno spopolnjevanje 2025

UVOD

- ogledali si bomo dvojno nihalo - preprost, a bogat sistem, ki povezuje fiziko, matematiko in teorijo kaosa.
- je klasičen primer, kako lahko majhne spremembe vodijo do nepredvidljivih rezultatov.
- obravnavali bomo njegovo mehaniko, enačbe, kaos in uporabo



matematično nihalo

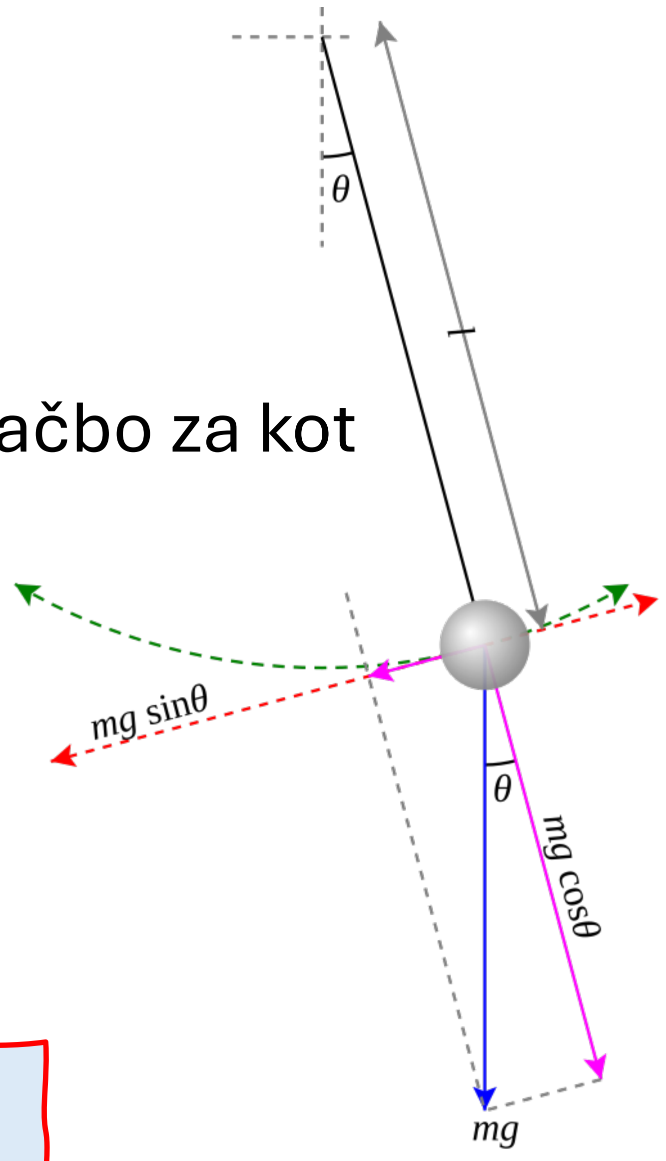
- skicirajte nihalo in označite relevantne količine
- zapišite Newtonov zakon in izpeljite dinamično enačbo za kot

$$F = -mg\sin\theta = ma,$$
$$a = -g\sin\theta,$$

$$s = \ell\theta,$$
$$v = \frac{ds}{dt} = \ell\frac{d\theta}{dt},$$
$$a = \frac{d^2s}{dt^2} = \ell\frac{d^2\theta}{dt^2},$$

$$\ell\frac{d^2\theta}{dt^2} = -g\sin\theta,$$
$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0.$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$



https://en.wikipedia.org/wiki/Pendulum_(mechanics)

matematično nihalo

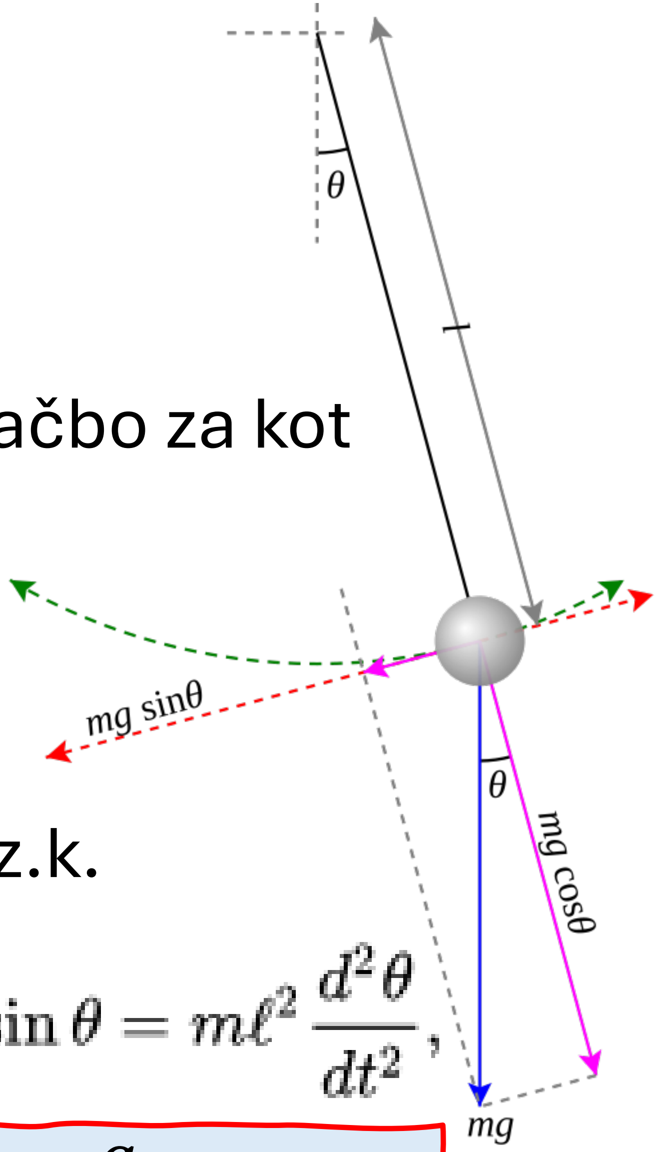
- skicirajte nihalo in označite relevantne količine
- zapišite Newtonov zakon in izpeljite dinamično enačbo za kot

$$F = -mg\sin\theta = ma, \quad a = -g\sin\theta,$$

so

$$s = \ell\theta, \quad v = \frac{ds}{dt} = \ell\frac{d\theta}{dt}, \quad a = \frac{d^2 s}{dt^2} = \ell\frac{d^2\theta}{dt^2},$$

$$\ell\frac{d^2\theta}{dt^2} = -g\sin\theta, \quad \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0.$$



- zapišite Newtonov zakon za vrtenje in izpeljite d.e.z.k.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \frac{d\mathbf{L}}{dt}. \quad |\boldsymbol{\tau}| = -mg\ell\sin\theta,$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad |\mathbf{L}| = mr^2\omega = m\ell^2\frac{d\theta}{dt}.$$

$$-mg\ell\sin\theta = m\ell^2\frac{d^2\theta}{dt^2},$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

matematično nihalo

- zapišite energijo nihala med nihanjem
- $E = mgh + \frac{1}{2}mv^2 = mgl(1 - \cos\theta) + \frac{1}{2}ml^2\dot{\theta}^2$
- $\dot{\theta} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$
- upoštevajte, da se pri nedušenem nihanju polna energija nihala ne spreminja, in izpeljite enačbo nihanja
- $\dot{E} = 0 \Rightarrow -mgl\,\dot{\theta}\sin\theta = ml^2\dot{\theta}\ddot{\theta}$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

Lagrangeov formalizem

- generalizirane koordinate
- fazni prostor
- Lagrangeova funkcija $L = T - V$

- načelo stacionarne akcije
  - akcija sistema $S[\boldsymbol{q}(t)] = \int_{t_1}^{t_2} L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) dt$
  - je v stacionarni točki

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

Euler-Lagrangeova enačba

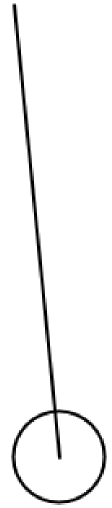
matematično nihalo - Lagrangeov formalizem

- generalizirane koordinate θ
- fazni prostor $q = \theta, \dot{q} = \dot{\theta}$
- Lagrangeova funkcija $L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl + mgl \cos\theta$
- Lagrangeova enačba $\frac{d}{dt}\left(ml^2\dot{\theta}\right) = -mgl \sin\theta$

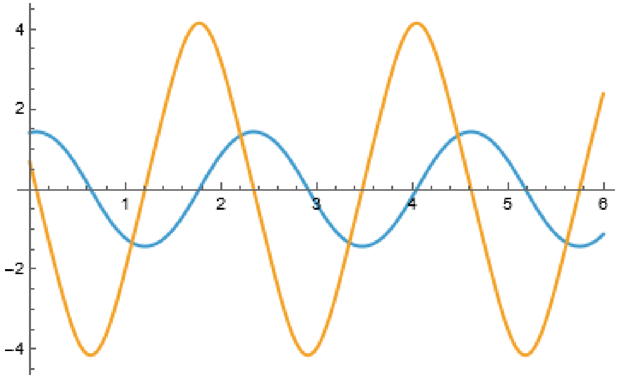
$$\ddot{\theta} = -\frac{g}{l} \sin\theta$$

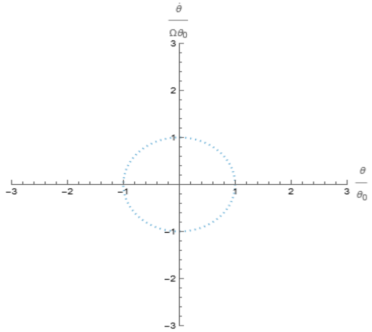
rešitev enačbe nihanja

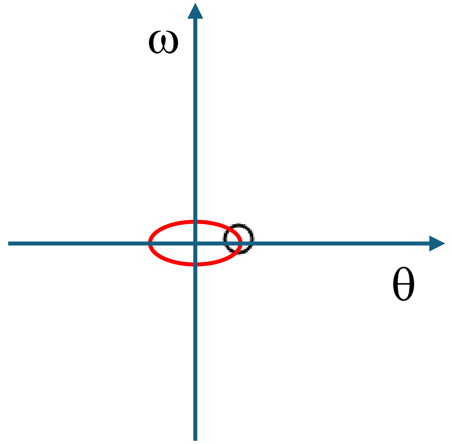
- $$\ddot{\theta} = -\frac{g}{l}\sin\theta$$



- majhni koti $\theta \doteq \sin\theta$
- $\ddot{\theta} = -\Omega^2\theta$
- $\theta = \theta_0 \cos(\Omega t + \delta)$
- $\dot{\theta} = -\Omega\theta_0 \sin(\Omega t + \delta)$

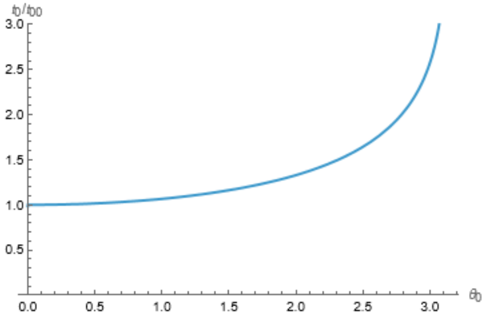


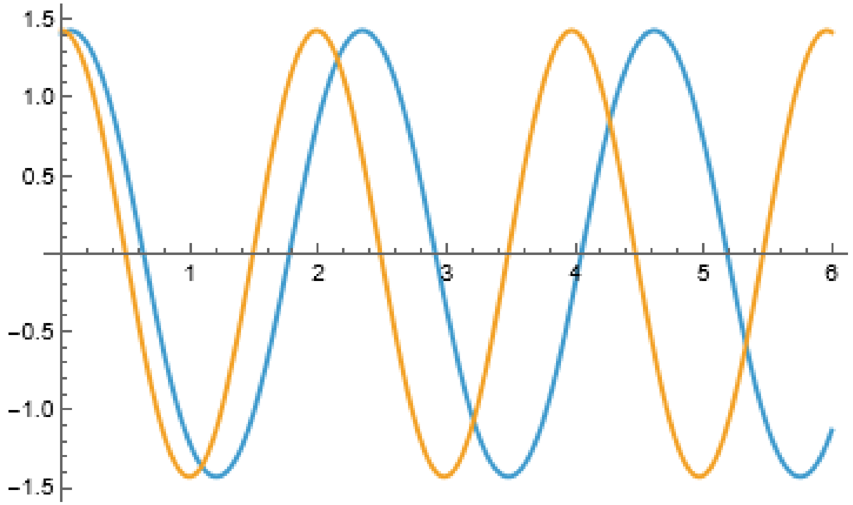




tirnica v faznem prostoru

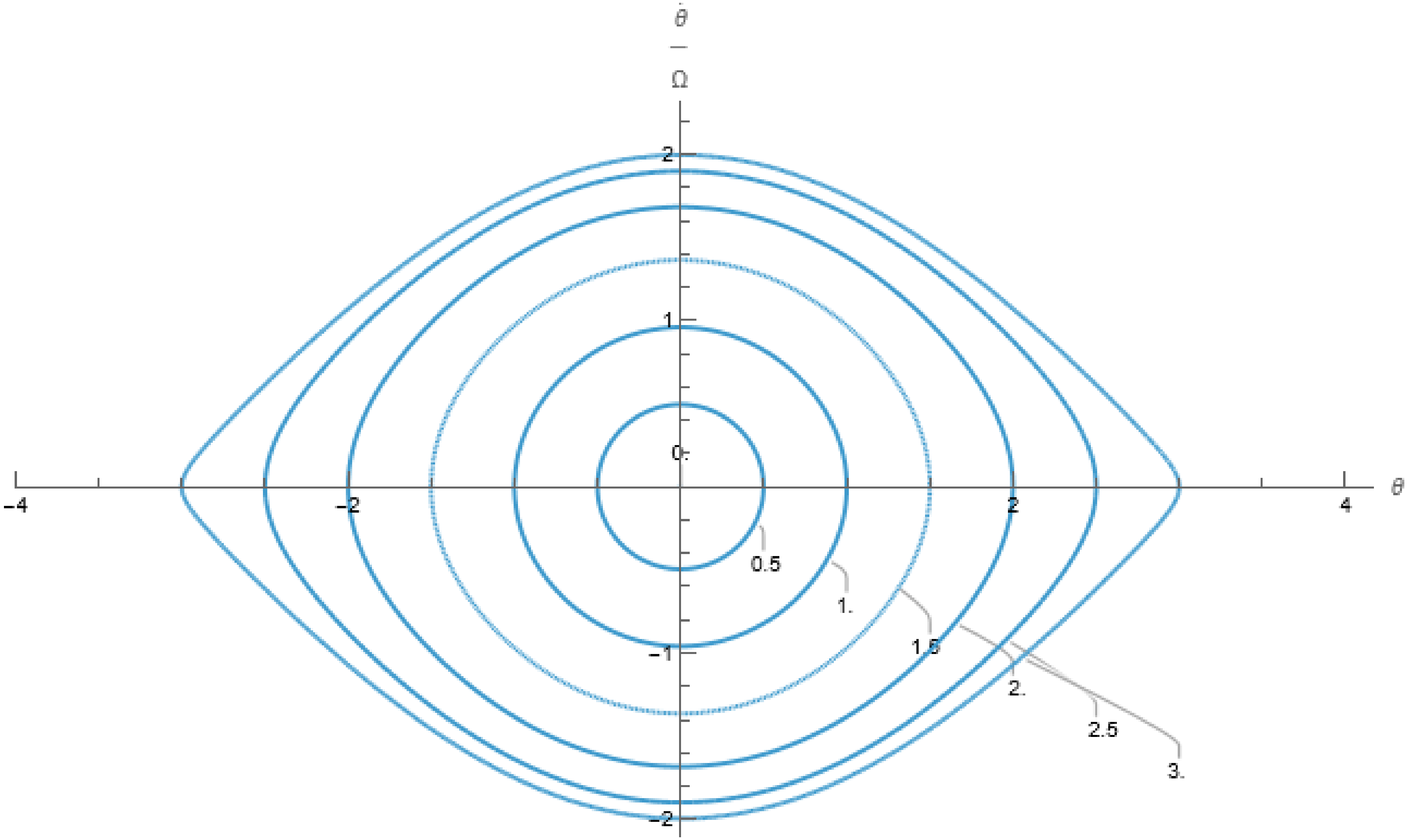
večji koti



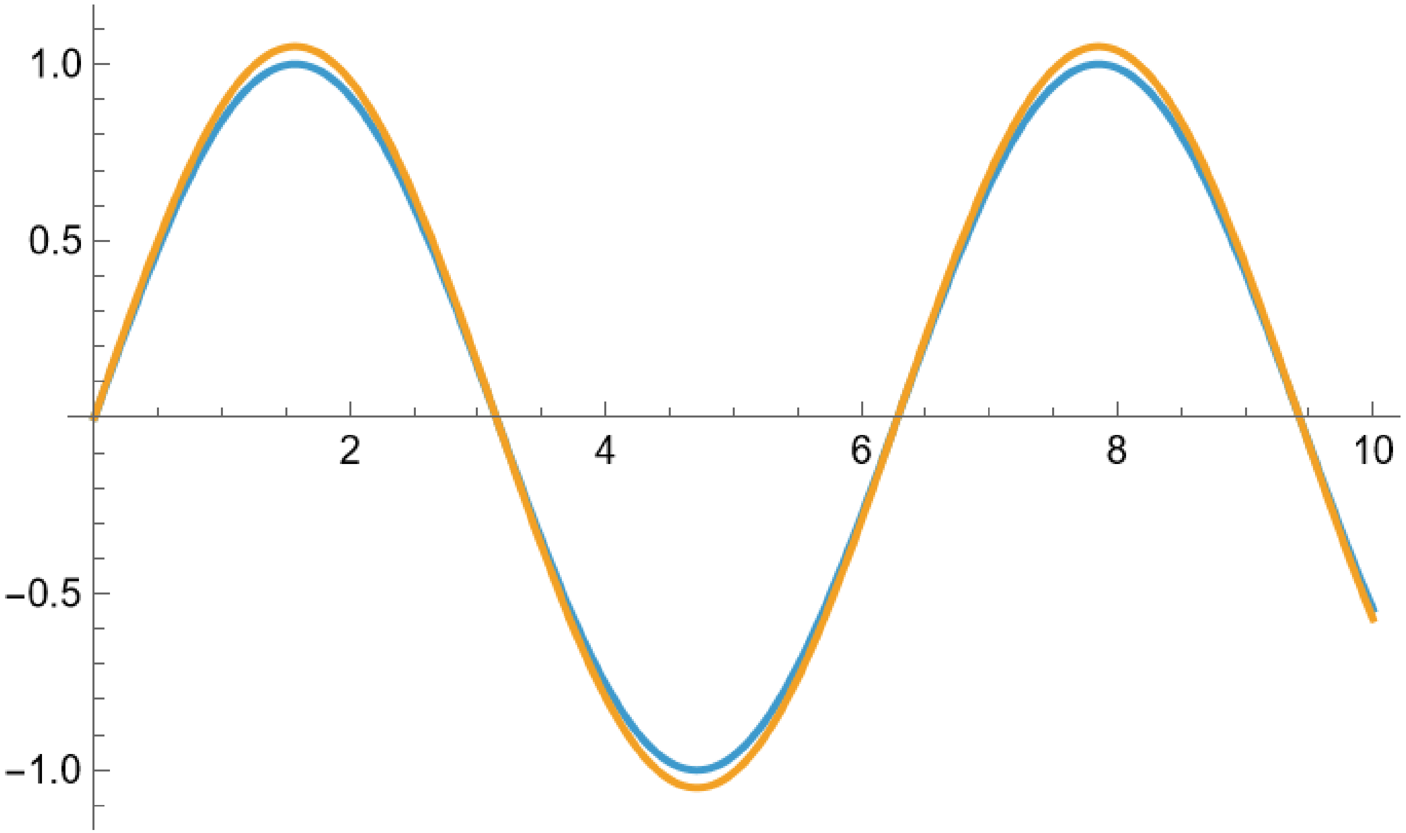


- $\dot{\theta} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$
- $\int_{\theta_0}^{\theta} \frac{d\theta\prime}{\sqrt{\frac{2g}{l}(\cos\theta\prime - \cos\theta_0)}} = \int_0^t dt'$
- $\int_{\theta_0}^{\theta} \frac{d\theta\prime}{\sqrt{\frac{4g}{l}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta'}{2}\right)}} = t$
- $\int_{\phi}^{\pi/2} \frac{d\phi\prime}{\sqrt{1-k^2\sin^2\phi\prime}} = \Omega t,\ k = \sin\frac{\theta_0}{2},\ \sin\frac{\theta'}{2} = k\sin\phi$
- $F\left(\frac{\pi}{2}, k\right) - F(\phi, k) = \Omega t,\ F(\phi, k)$ je nepopolni eliptični integral prve vrste
- $\phi = \mathrm{am}(K(k) - \Omega t, k)$, am() je Jacobijeva amplitudna funkcija
- $\theta(t) = 2\arcsin\left(k\sin\left(\mathrm{am}(K(k) - \Omega t, k)\right)\right)$

razvoj sistema v faznem prostoru

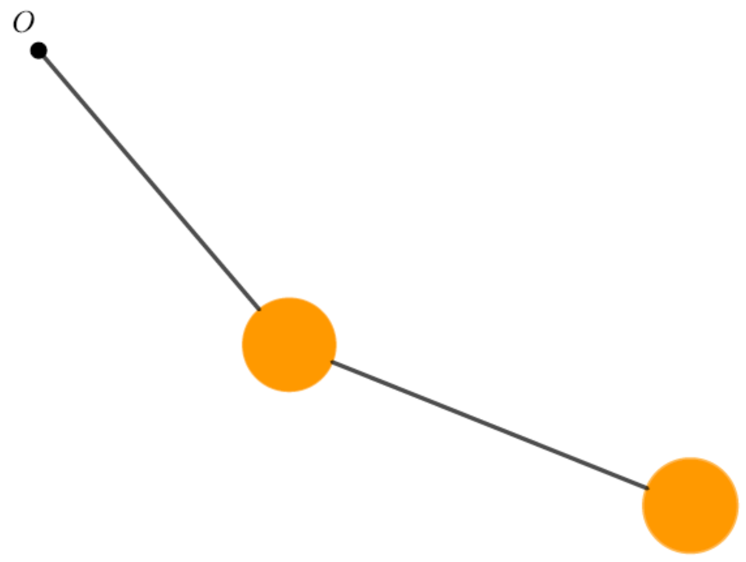


pohleven sistem



dvojno nihalo

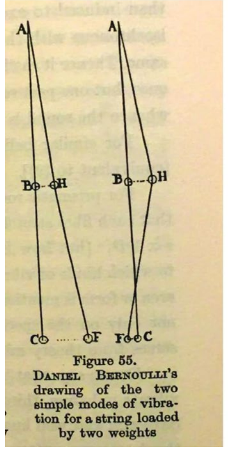
- sestavljeno je iz dveh (nitnih – lahko fizičnih) nihal, eno je pritrjeno v fiksno osišče *O*, drugo visi na prvem
- sestavni deli: dve masi (m_1, m_2), dve palici (dolžine l_1, l_2), osišče in gravitacija
- za razliko od enega nihala njegovo gibanje ni (le) periodično



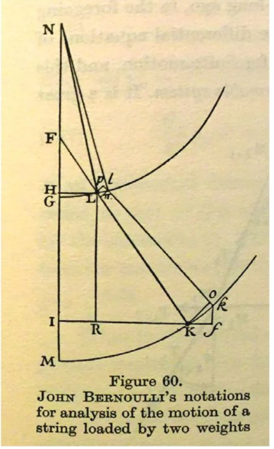
zgodovinski pregled

- 1738 Euler, Bernoulli
- 1788: Lagrangeova mehanika,
- 1890: Poincaréjevi vpogledi v kaos,

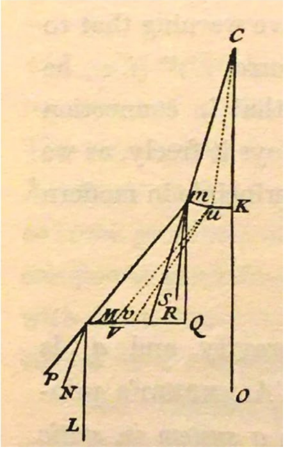
Daniel Bernoulli
(1733)

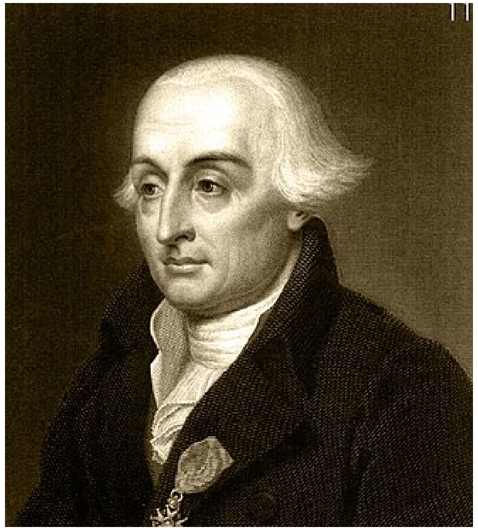


Johann I Bernoulli
(1742)



D'Alembert
(1743)

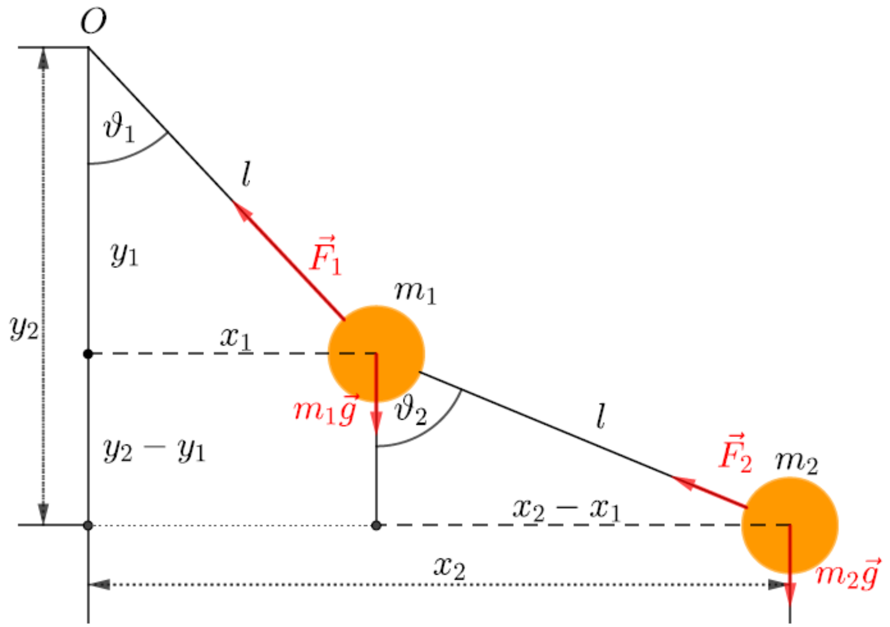




https://galileo-unbound.blog/tag/double-pendulum/

fizika nihala

- gravitacija poganja gibanje, medtem ko ga napetost v palicah omejuje
- opišeta ga dva kota (θ_1, θ_2)
- kinetična in potencialna energija se izmenjujeta med nihanjem



Lagrangeov formalizem

$$x_1 = l_1 \sin\theta_1$$
$$y_1 = -l_1 \cos\theta_1$$
$$x_2 = x_1 + l_2 \sin\theta_2$$
$$y_2 = y_1 - l_2 \cos\theta_2$$

$$\dot{x}_1 = l_1\dot{\theta}_1 \cos\theta_1$$
$$\dot{y}_1 = l_1\dot{\theta}_1 \sin\theta_1$$
$$\dot{x}_2 = l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2$$
$$\dot{y}_2 = l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2 \sin\theta_2$$

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+g(m_1 + m_2)l_1 \cos\theta_1 + g m_2 l_2 \cos\theta_2$$

Lagrangeov formalizem

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{\partial L}{\partial \theta_1} \wedge \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{\partial L}{\partial \theta_2}$$

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin\theta_1 = 0$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0$$

Newtonov formalizem

$$ma_{x1} = F_{x1} = -F_1 \sin\vartheta_1 + F_2 \sin\vartheta_2\,,$$

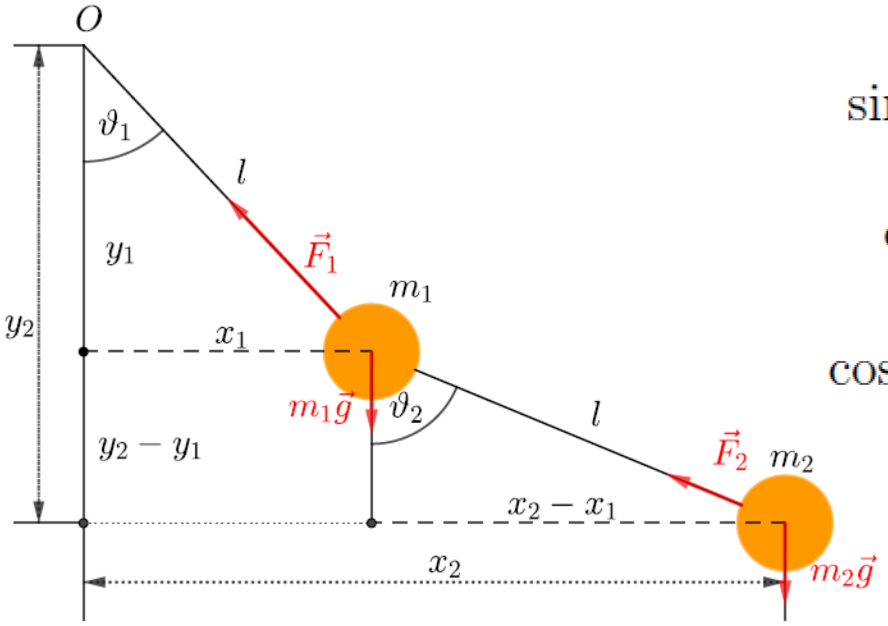
$$ma_{x2} = F_{x2} = -F_2 \sin\vartheta_2\,,$$

$$ma_{y1} = F_{y1} = -mg + F_1 \cos\vartheta_1 - F_2 \cos\vartheta_2\,,$$

$$ma_{y2} = F_{y2} = -mg + F_2 \cos\vartheta_2\,,$$

$$F_1 = k\left(l - \sqrt{x_1^2 + y_1^2}\right)\,,$$

$$F_2 = k\left(l - \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)\,.$$



$$\sin\vartheta_1 = \frac{x_1}{l}\,,$$

$$\sin\vartheta_2 = \frac{x_2 - x_1}{l}\,,$$

$$\cos\vartheta_1 = -\frac{y_1}{l}\,,$$

$$\cos\vartheta_2 = -\frac{y_2 - y_1}{l}\,.$$

numerična simulacija

$$v_{x1,n+1} = v_{x1,n} + (-f_{1,n}\sin\vartheta_{1,n} + f_{2,n}\sin\vartheta_{2,n})\,\Delta t\,,$$

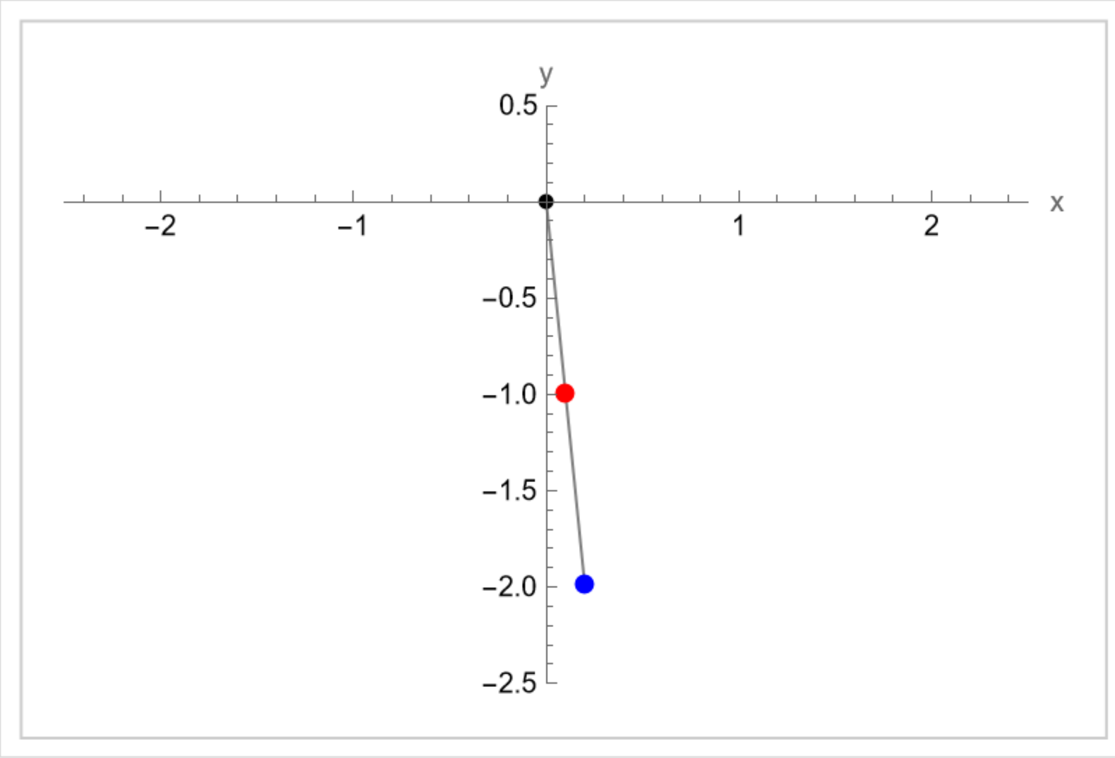
$$v_{y1,n+1} = v_{y1,n} + (-g - f_{1,n}\cos\vartheta_{1,n} - f_{2,n}\cos\vartheta_{2,n})\,\Delta t\,;$$

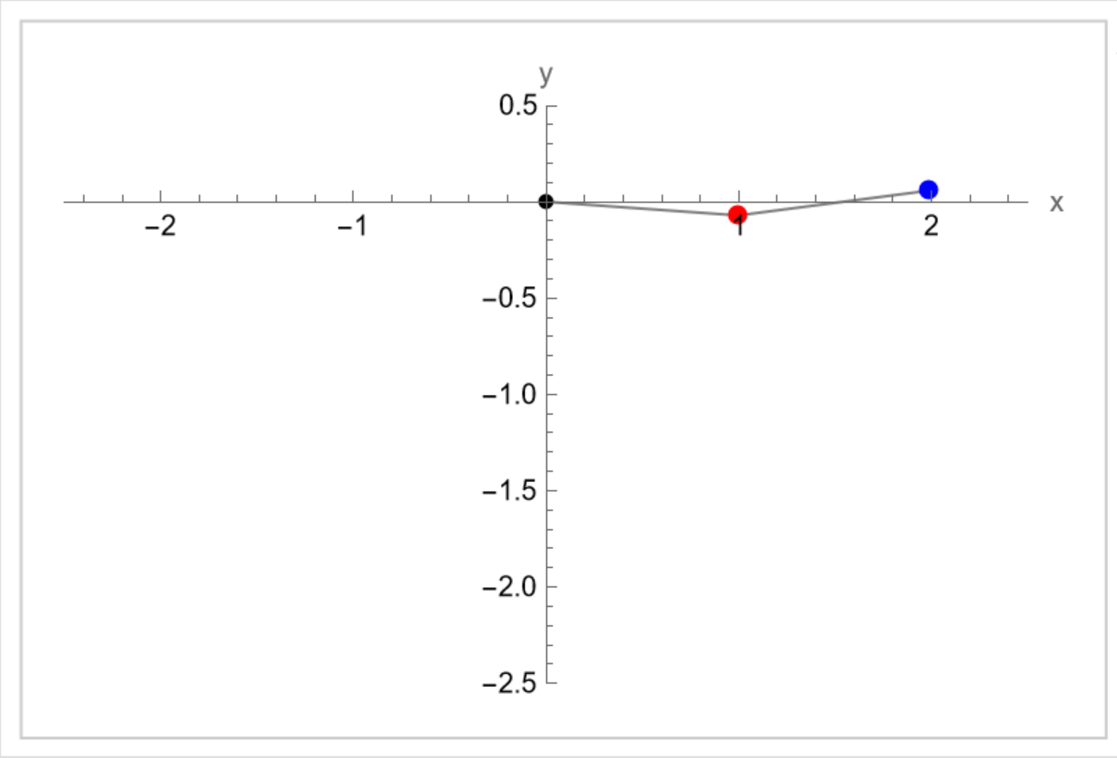
$$v_{x2,n+1} = v_{x2,n} + (-f_{1,n}\sin\vartheta_{2,n})\,\Delta t\,,$$

$$v_{y2,n+1} = v_{y2,n} + (-g + f_{2,n}\cos\vartheta_{2,n})\,\Delta t\,,$$

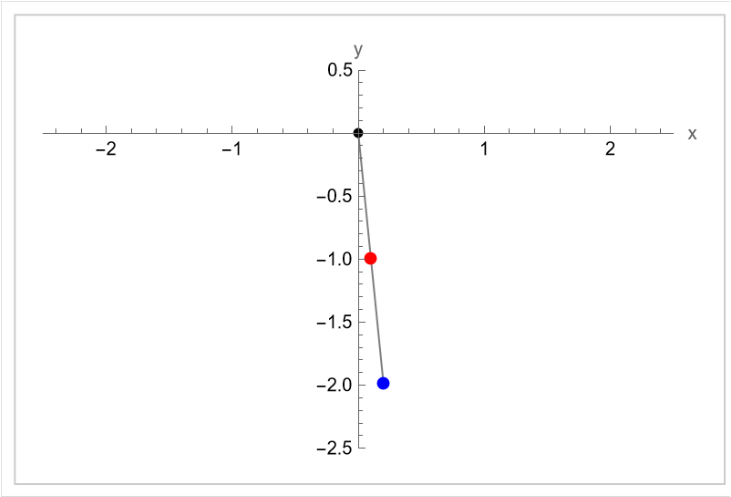
$$x_{1,n+1} = x_{1,n} + v_{x1,n+1}\Delta t\,.$$

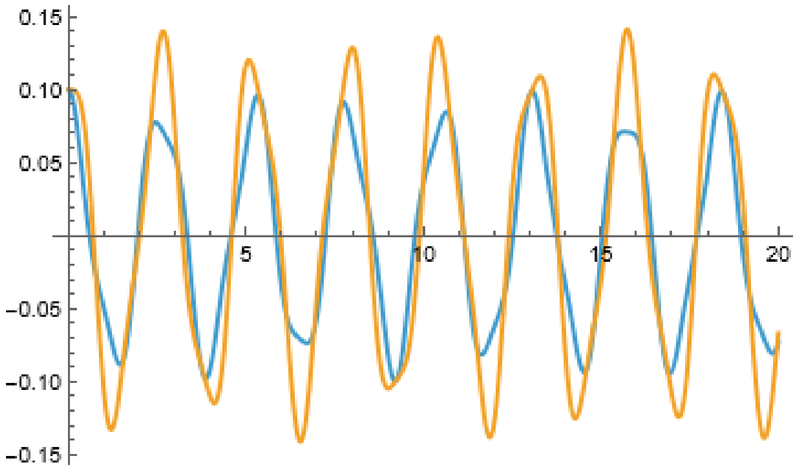
predstavitev gibanja?

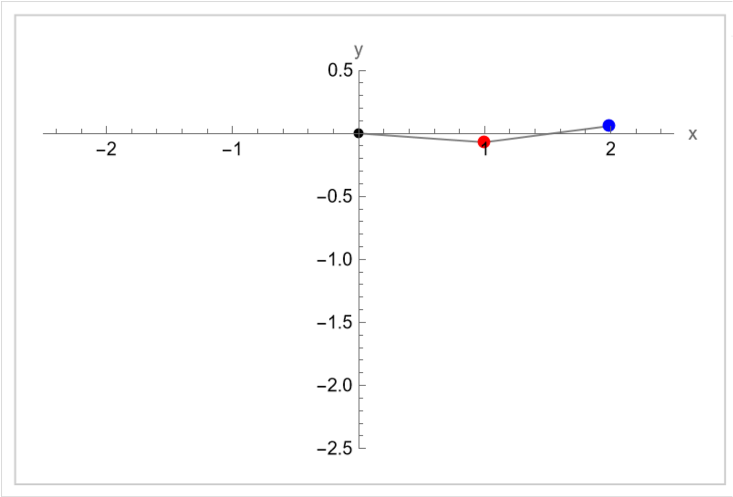


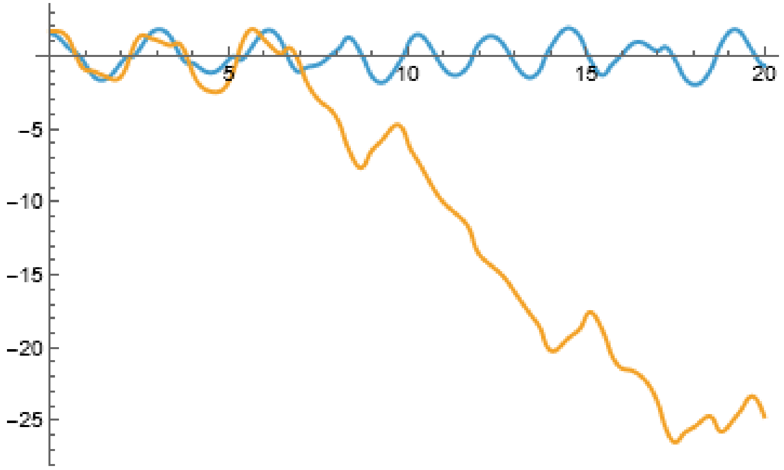


predstavitev gibanja?

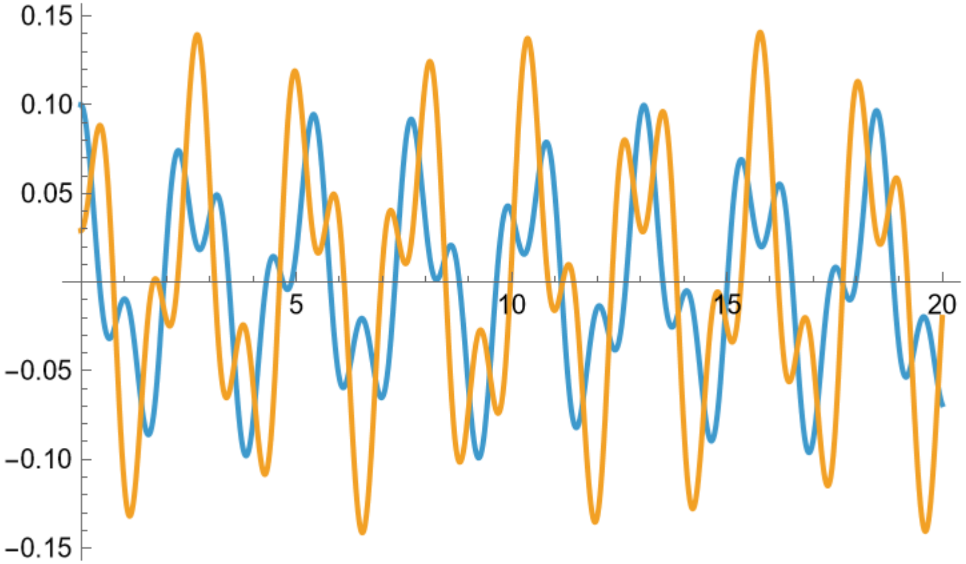






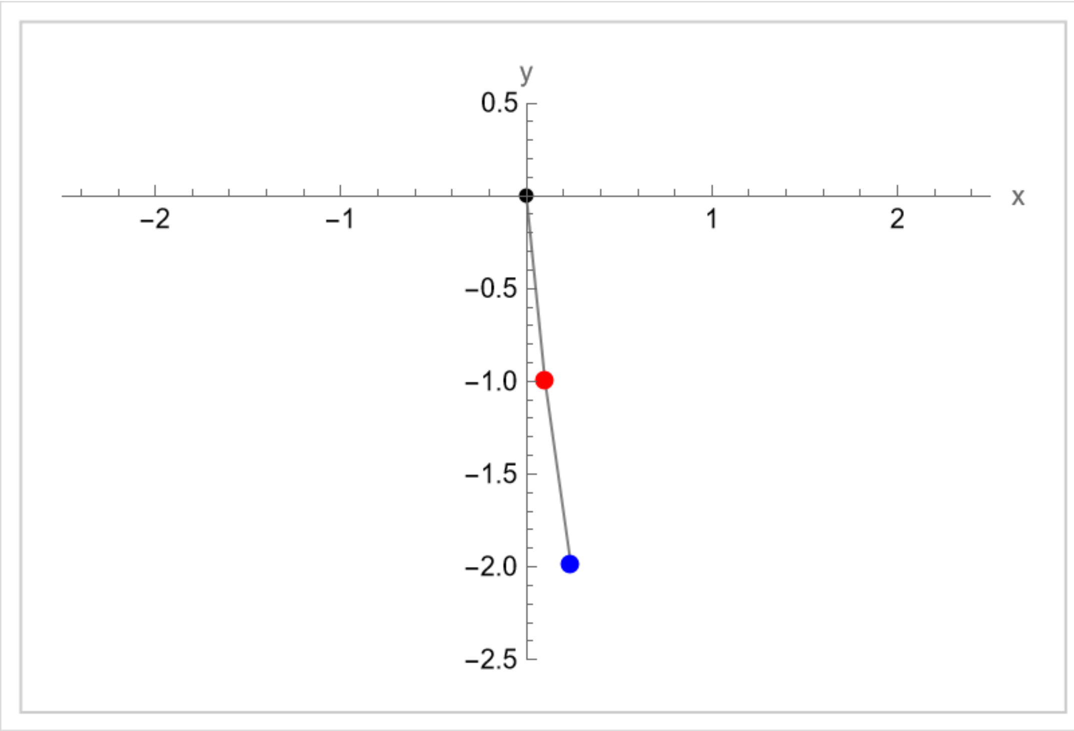


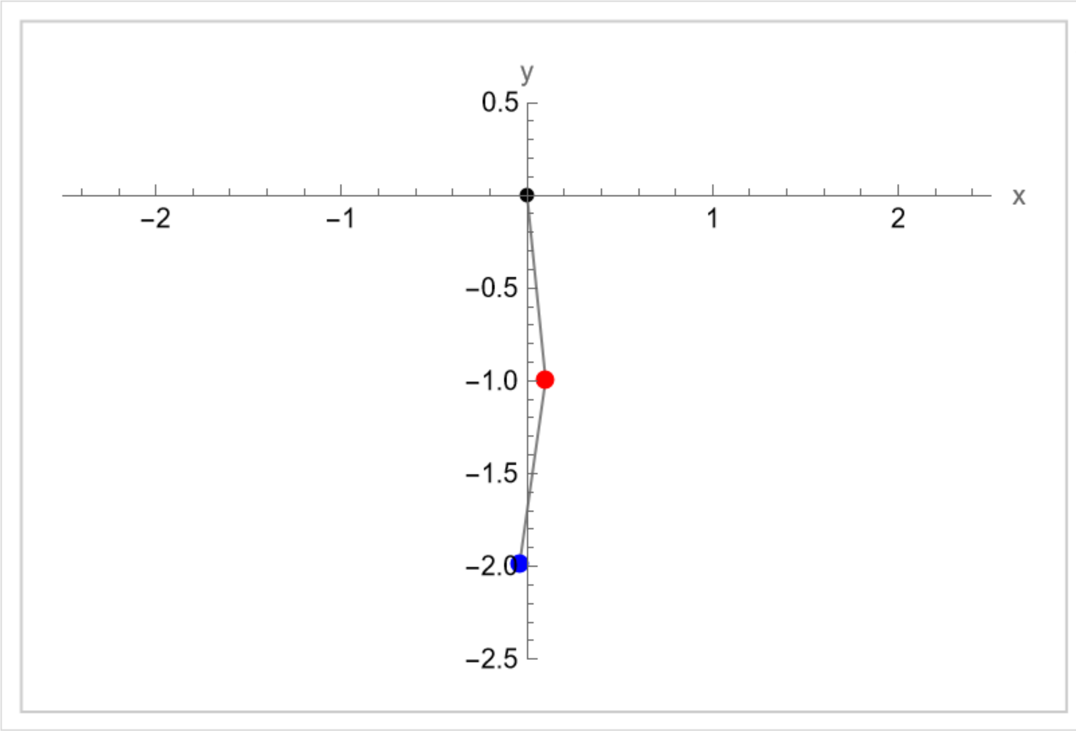
majhne amplitude



lastna.cdf

lastna nihanja dvojnega nihala





- Newtonov zakon za uteži:
- $ma_{x1} = -2mg\frac{x_1}{l} + mg\frac{x_2 - x_1}{l}$
- $ma_{x2} = -mg\frac{x_2 - x_1}{l}$

- harmonični nastavek za odmika
- $a_{x1,2} = -\omega^2 x_{1,2}$

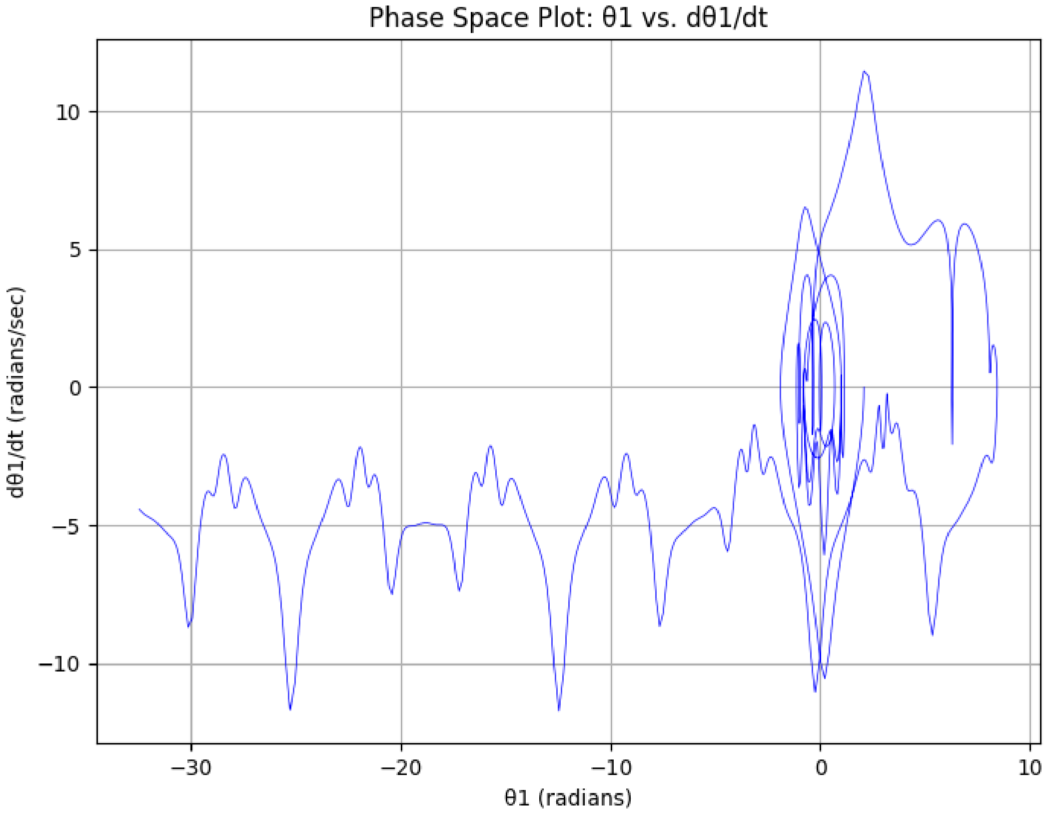
- $-\omega^2 x_{01}\cos\omega t = -2g\frac{x_{01}}{l}\cos\omega t + g\frac{x_{02} - x_{01}}{l}\cos\omega t$
- $-\omega^2 x_{02}\cos\omega t = -g\frac{x_{02} - x_{01}}{l}\cos\omega t$

- $\left(-\omega^2 + 3\frac{g}{l}\right)x_{01} = \frac{g}{l}x_{02}$
- $\frac{g}{l}x_{01} = \left(-\omega^2 + \frac{g}{l}\right)x_{02}$

- izrazimo x_{01}/x_{02} , izenačimo: $\omega^4 - 4\frac{g}{l}\omega^2 + 2\frac{g^2}{l^2} = 0$

večje amplitude - kaos

- kaos = občutljivost na začetne pogoje + nepredvidljiv dolgoročni potek
- tudi s popolnimi enačbami napovedovanje po kratkem času odpove
- Lyapunov eksponent meri, kako hitro se trajektorije razhajajo - pozitiven za kaos

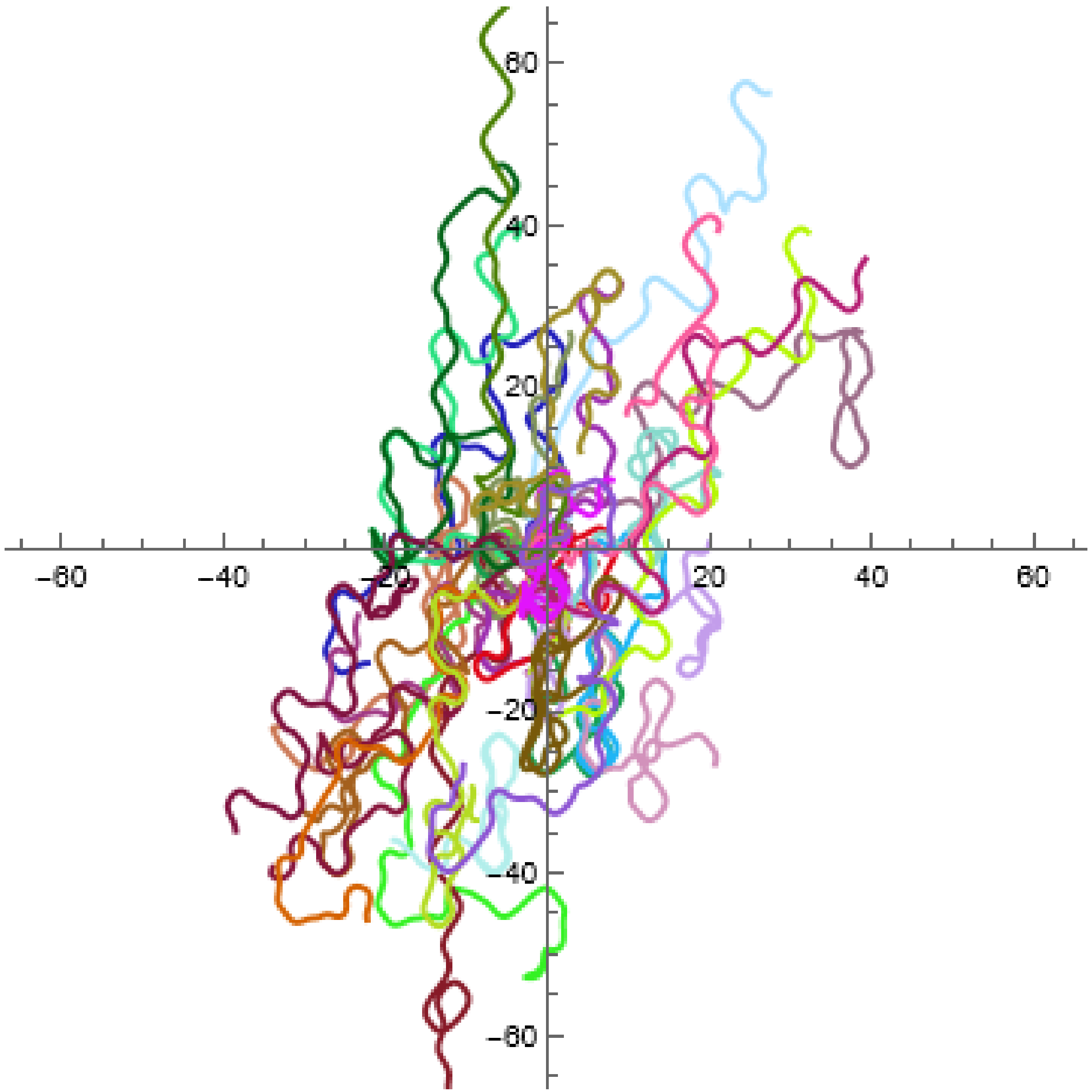


vizualizacija kaosa

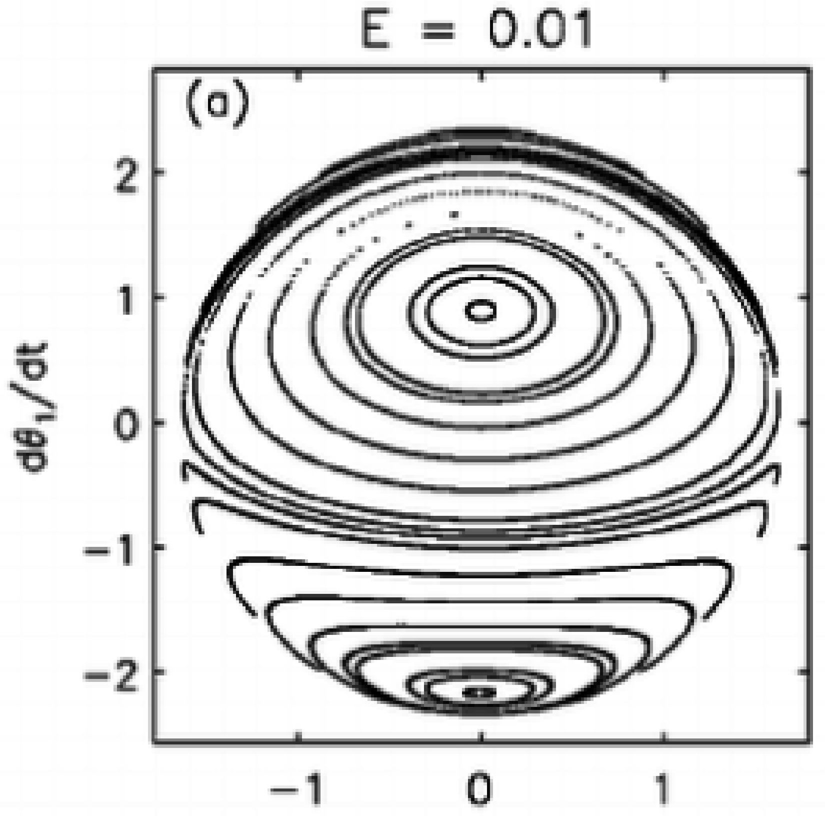
- Začne se urejeno, nato postane nepredvidljivo - obračanje, zanka
- za 0,01 ° različni začetni pogoji se razidejo v sekundah.

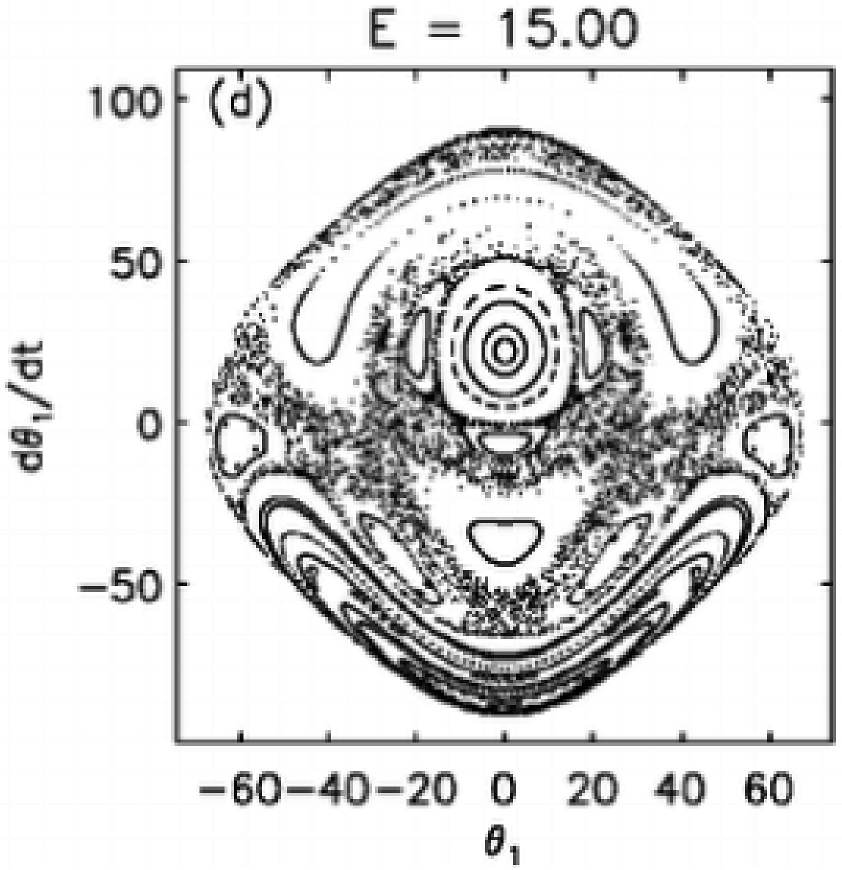
difuzija

- theta1 = Pi/1.2
- theta2 = Pi/f in {f, 1.1, 1.4, 0.01}

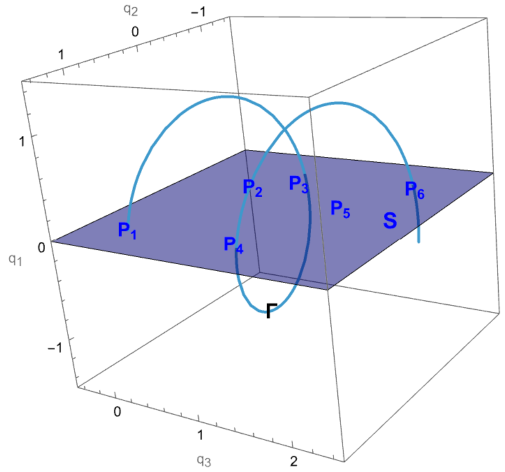


Poincarejevi zemljevidi

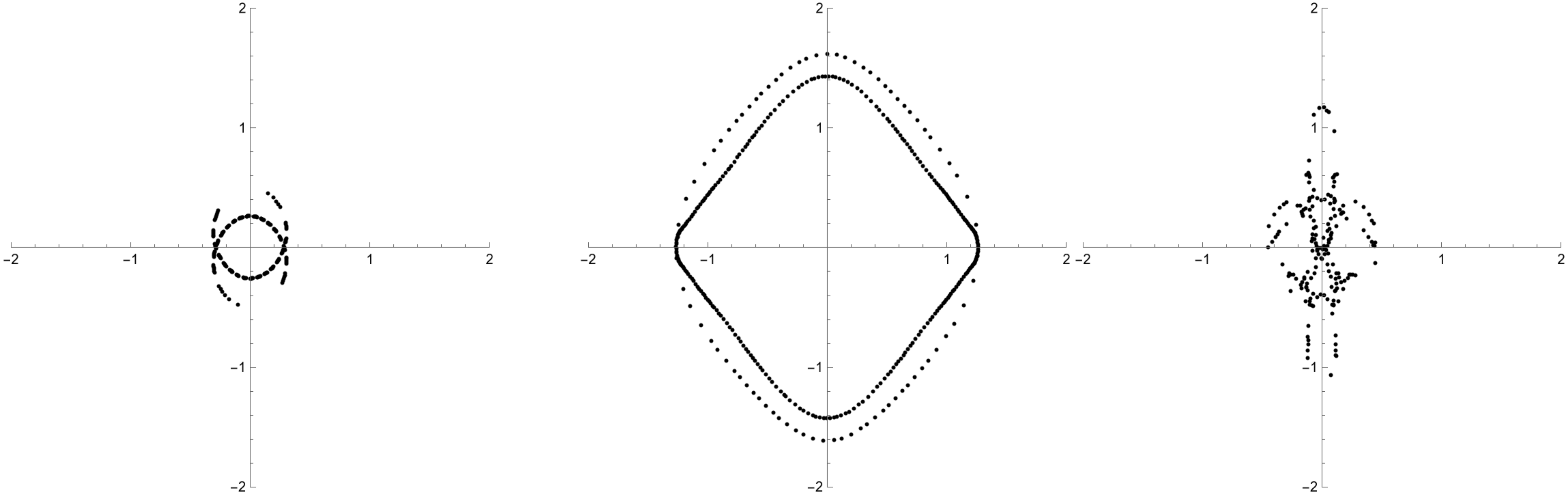




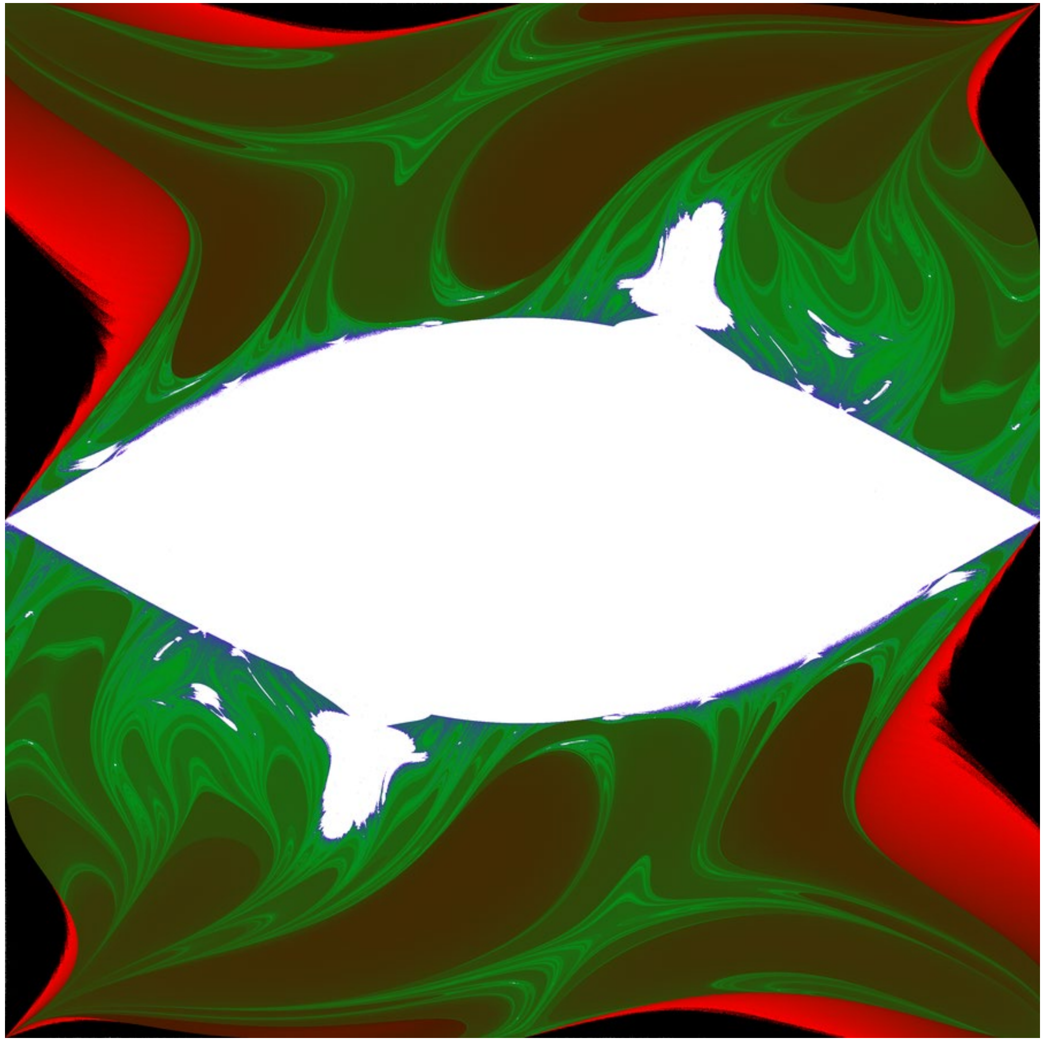
- $\frac{d\boldsymbol{q}}{dt} = F(\boldsymbol{q}, t)$ lahko razumemo kot tok v faznem prostoru (tokovnica) – težava z dimenzijo prostora



- Gama je tok, preslikava iz zveznega toka v diskretni prostor poinc.cdf

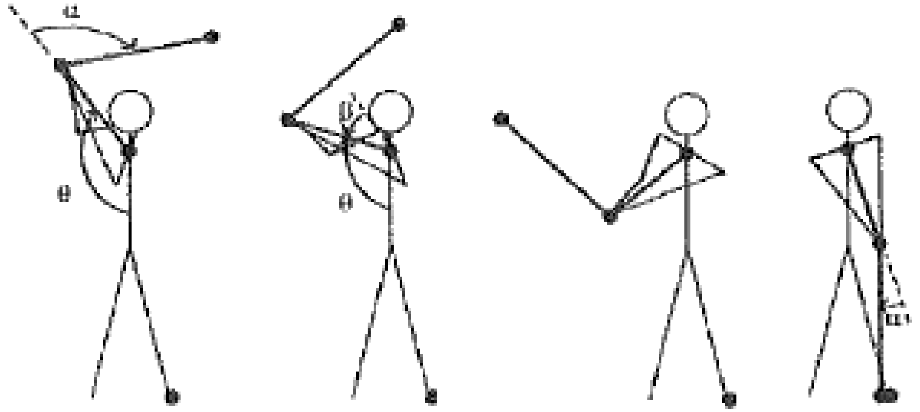


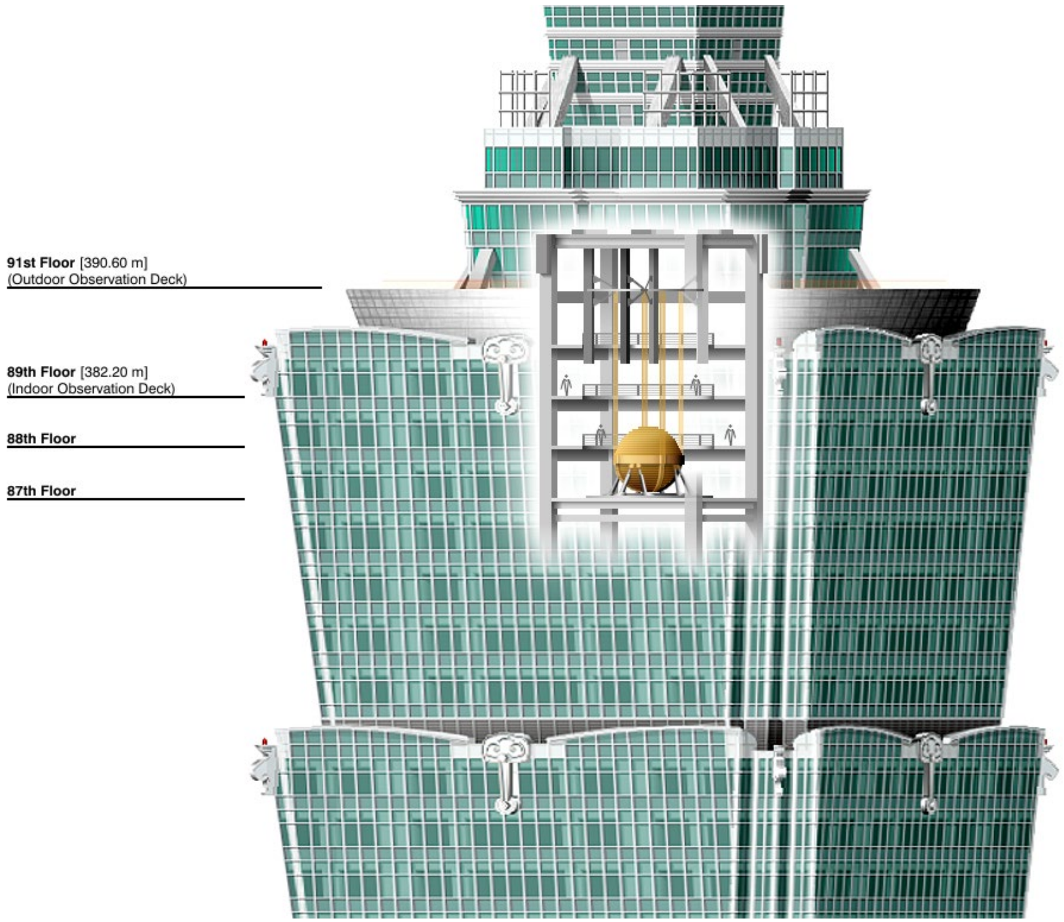
fraktali



FurtadoG, CC https://commons.wikimedia.org/wiki/File:Double_pendulum_flip_time_2021.png

uporaba




91st Floor [390.60 m]
(Outdoor Observation Deck)
89th Floor [382.20 m]
(Indoor Observation Deck)
88th Floor
87th Floor

SIMULACIJE

- https://www.myphysicslab.com/pendulum/double-pendulum-en.html