Teaching E=mc^2: An exploration of some issues

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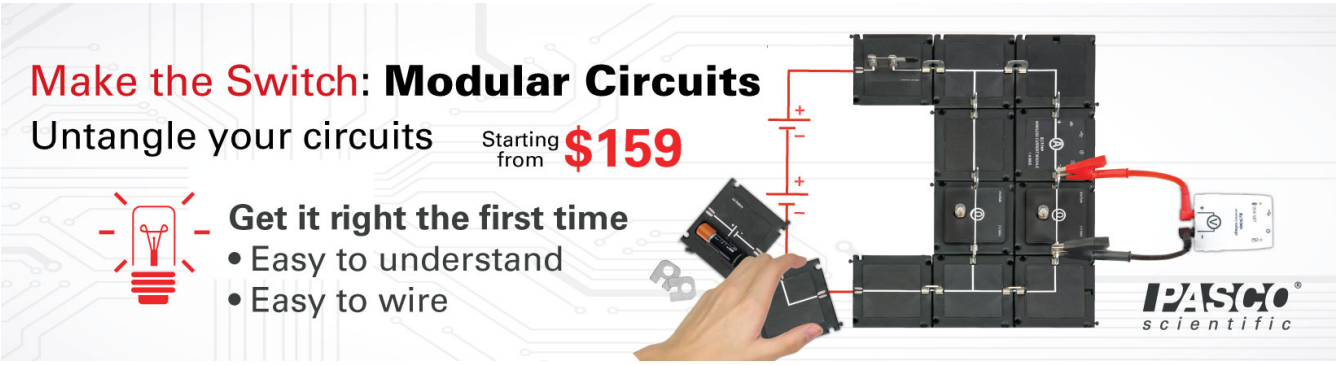
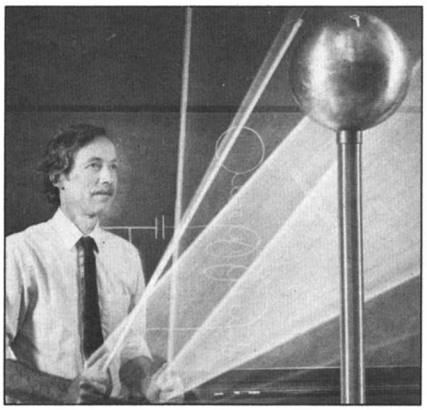


Teaching $E = mc^2$:

By Ralph Baierlein



Ralph Baierlein *received his Ph.D. from Princeton, taught at Harvard, and then—in 1966—joined the faculty of Wesleyan University (Middletown, CT 06457). He serves as book review editor for the* American Journal of Physics *and, as an astrophysicist, studies star formation. Teaching physics to nonscience students is one of his favorite occupations. This article grew out of a course for those students.*

Everyone knows the equation $E = mc^2$, but what does it mean? This article addresses the question and explores some other issues as well.

Because the equation links energy and mass, we need to be clear about what those words mean. In physics, *energy* means the ability to do work, for example, to lift a weight. The word *mass* denotes inertia, a reluctance to undergo a change in velocity. (This is not the whole story about "mass" or "energy," but it suffices for now. A lot more will be said later about both mass and energy.) A discussion (later on) of certain distinctions will require the word "matter." *Matter* is tangible stuff, what you can hold in your hand.

Matter, mass, and energy: in what sense do they *exist*? Take a dime out of your pocket and flip it into the air. The dime is an example of matter; matter, we see, exists as a *thing*. Traveling through the air, the dime has some inertia—it is not blown around by every little breeze—and it has energy, partly kinetic energy, partly gravitational potential energy. Thus mass and energy exist as *attributes* of a physical system, just as do color and shape. Mass and energy exist in the same fashion, and that fashion is different from the way matter exists.

A Derivation

A derivation of $E = mc^2$ will help us see the meaning we should attach to the equation. The neatest derivation that I know of is a variant of one of Albert Einstein's later derivations.[1,2]

Figure 1 shows Bob and Alice (who moves relative to Bob with speed v) and an atom initially at rest in Alice's reference frame. The atom emits two photons of equal energy, and we specify that, in Alice's frame, the two photons travel back-to-back and perpendicular to the direction of the frames' relative motion. Symmetry alone requires that the atom remain at rest in Alice's frame, and therefore the atom does *not change* its velocity in Bob's frame.

As observed from Bob's frame, the photons move along diagonal directions. The photons must keep up with the atom's forward motion (because, as observed by Alice, the two photons and the atom lie along a straight line), and so each photon's component of velocity parallel to the atom's velocity must be v. Thus the photons' momentum parallel to the atom's velocity is

$$\begin{pmatrix}\text{Number}\\ \text{of photons}\end{pmatrix}\begin{pmatrix}\text{entire momentum}\\ \text{of one photon}\end{pmatrix}\begin{pmatrix}\text{fraction that}\\ \text{is parallel}\end{pmatrix} = 2\,\frac{hf}{c}\,\frac{v}{c} \qquad (1)$$

Already James Clerk Maxwell calculated that the ratio of momentum to energy for a burst of classical electromagnetic radiation is 1/(speed of light). Therefore a photon of energy hf has a momentum of magnitude hf/c; here f denotes the frequency of the light wave, and h, Planck's constant. (Actually, most of my students learn about this momentum-to-energy ratio for electromagnetic radiation when they study the Compton effect.)

Now we examine energy and momentum changes for the atom as observed by Bob. Because the atom emits two photons, each of energy hf, energy conservation implies that the atom's energy decreases:

An Exploration of Some Issues

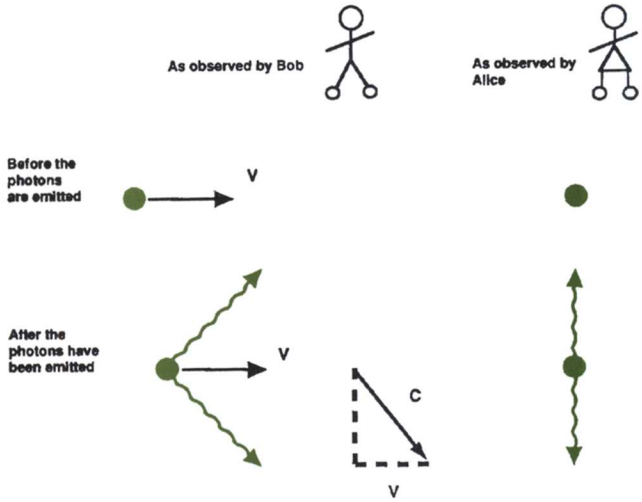


Fig. 1. The emission of two photons as observed by Alice and Bob. The small triangle illustrates that each photon's component of velocity parallel to the atom's velocity has the value *v*.

$$\Delta\, Energy_{atom} = -2hf \tag{2}$$

Conservation of momentum implies that the *change* in the atom's momentum must be equal in magnitude to the momentum of the photons but opposite in direction. Thus

$$\Delta\, momentum_{atom} = -2\,\frac{hf}{c}\,\frac{\vec{v}}{c} = \Delta\, Energy_{atom}\,\frac{\vec{v}}{c^2} \tag{3}$$

The second step follows by substitution for $-2hf$ from above.

The definition of momentum (as mass times velocity) implies

$$\Delta\, momentum_{atom} = (\Delta\, mass_{atom})\,\vec{v} \tag{4}$$

because the atom's velocity remains constant in both frames.

The two expressions for the change in the atom's momentum must agree, and so we deduce that

$$\Delta\, Energy_{atom}\,\frac{\vec{v}}{c^2} = (\Delta\, mass_{atom})\,\vec{v} \tag{5}$$

After multiplying on both sides by c^2 and recognizing that the coefficients of $\vec{v}$ must be equal, we emerge with

$$\Delta E = (\Delta m)c^2 \tag{6}$$

where E and m denote the atom's energy and mass, respectively. In words, when the energy of the atom decreased, so did its inertia.

Note that the atom has just as many electrons, protons, and neutrons as before. Only the electrical potential energy and the kinetic energy of the electrons have changed. (The emission of light—the two photons—is accompanied by a change in how the electrons swarm around the nucleus, and that change alters the electrical potential energy and the kinetic energy.) The change in the atom's energy is accompanied, however, by a change in the atom's inertia. When the energy decreased, so did the inertia.

We now have a connection between ΔE and Δm, between changes in energy and in mass. We have not yet arrived at $E = mc^2$, but we are close.

Moreover, already we can make the point that one must distinguish *conversions* from *parallel changes*. One can convert potential energy to kinetic energy and vice versa. Similarly, one can convert matter to radiation (as in electron-positron annihilation) and radiation to matter (as in pair production). With mass and energy, the situation is radically different: changes in mass and energy occur *in parallel*. There is *no conversion* of one into the other. As Einstein noted in his first paper[3] on the subject, "if the energy changes by ΔE, then the mass changes in the same sense by $\Delta E/c^2$."

The notion of parallel changes may be easier to accept if we remember that mass means inertia and is an attribute. For an atom or any object, the attributes energy and inertia change in parallel.

To arrive at $E = mc^2$, we can imagine creating an object (along with its anti-object) out of radiation. Every bit of energy that goes into creating the object would be accompanied by an increment in mass; according to Eq. (6), we have $\Delta E = (\Delta m)c^2$. Starting from zero for *both* the object's energy *and* its mass and adding up increments, we emerge with

$$E = mc^2 \tag{7}$$

Einstein saw the equation $E = mc^2$ as the most significant single result of the special theory of relativity. Energy and inertia had been distinct notions—and to some extent they still are—but here is a synthesis. The attribute "energy" is always accompanied by the attribute "inertia"; the propor-

tionality constant is c^2. We abuse the language only slightly if we say that inertia is a property of energy.

A few more words about the present derivation of $\Delta E = (\Delta m)c^2$ are in order. I used two photons, but two bursts of classical electromagnetic radiation would serve just as well. Indeed, such bursts are what Eric Rogers[1] and Einstein[2] used. The derivation requires only the ratio of momentum to energy for electromagnetic radiation; that ratio is simply $1/c$, and thus quantum theory is not essential. Moreover, this derivation—unlike Einstein's 1905 derivation—makes no use of Lorentz transformations or other results from the special theory of relativity. In short, by 1873, Maxwell knew everything necessary to derive the equation $\Delta E = (\Delta m)c^2$. All that was missing was a context of inquiry that would have led him to search for a connection between energy and inertia.

Changes in mass and energy occur in parallel. There is no conversion of one into the other.

In the derivation of $E = mc^2$ and in the discussion so far, I have used the word "inertia" frequently. I did this to emphasize that the symbol m in mc^2 denotes "mass" in the sense of "inertia." Most certainly, the m does *not* mean "matter."

While we are looking at the issues, let me address a few other points.

Mass in History

The word "mass" has a long history and a correspondingly long string of meanings and nuances. Max Jammer provides a fine discourse on the topic in his *Concepts of Mass*,[4] on which I rely for the early history. In Roman times, the Latin word "massa" denoted a lump of dough or paste. This sense of "lump of stuff" is what we have in mind when we "hang a mass from a spring." By Isaac Newton's time, the word "massa" (still in Latin, of course) had come to denote "quantity of matter," a valuable concept but one that defied attempts to make it both operationally precise and independent of any prior notion of density.

Newton recognized clearly the physical notion of inertia and hence what we would today call "inertial mass." It was Leonhard Euler, however, who first defined inertial mass operationally as the ratio of force to acceleration (really, as the ratio of their magnitudes). The constancy of the ratio (at least within the experience of seventeenth- and eighteenth-century physics) provided a characteristic value for each specific body: its inertial mass.

When Einstein was striving to develop his theory of gravity—the decade 1905 to 1915—physics recognized three distinct but related notions of "mass" for a specific body: its inertial mass, its active gravitational mass (meaning the extent to which the body produces a gravitational field), and its passive gravitational mass (meaning the extent to which the body is pulled on gravitationally by other bodies and hence a quantity proportional to the body's "weight"). Remarkably, all three of these "masses" are universally proportional to one another, a property already recognized by Newton.

To recapitulate, in the history of physics we have at least five distinct notions of "mass": lump of stuff, quantity of matter, inertial mass, active gravitational mass, and passive gravitational mass. (No wonder students are often confused.) For the narrow purpose of discussing the equation $E = mc^2$, however, we need only one notion of "mass": inertial mass. To put it positively, the "mass" that appears in $E = mc^2$ is the inertial mass. The history of the equation itself leaves no room for doubt on that score.[5] Einstein and others spoke explicitly about "inertia" [the German word being "Trägheit"] and "inertial mass" ["träge Masse"]. Indeed, Einstein's first paper[3] on the subject had for its title the question, "Does the inertia of a body depend on its energy content?"

Only Semantics

Even after one has agreed that in our context "mass" means "inertial mass" and hence "inertia," further specification is required. Let me set the scene with an example.

When an electron, say, moves at a substantial fraction of the speed of light, its momentum is related to its velocity by the equation

$$momentum = m_0 \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \vec{v} \tag{8}$$

where the symbol m_0 denotes the electron's inertia when it has zero speed. What should one call here the electron's "mass"? Three options arise:

1) One can call the entire coefficient of $\vec{v}$ the "mass," in which case the constant m_0 acquires the name "rest mass."
2) One can call m_0 the "mass" (leaving the square-root factor to convert simple $\vec{v}$ into part of the relativistic velocity four-vector).
3) One can turn to an entirely different expression and use it to define what one will call the electron's "mass":

$$\text{"mass"} = [E^2 - (momentum)^2 c^2]^{1/2}/c^2 \tag{9}$$

where E denotes the electron's energy.

There is no consensus on which option to adopt, and my use of the verb "call" indicates that the issue is one of semantics. Each option has its advantages and disadvantages. So long as people are clear that they are just making a choice of definition, it does not matter which option they choose.

In the derivation earlier in this article, I adopted option 1 implicitly. It provides the greatest continuity with the momentum and velocity relation in Newtonian physics, and—as we saw—it leads to $E = mc^2$. A disadvantage is that one needs to be circumspect in relating mass to force and acceleration. An operational definition of mass as the ratio of force to

acceleration (in magnitudes) requires the stipulation that the context be one of constant speed (as in magnetic deflection). This is reasonable, for if "mass" may depend on speed, then one ought to measure it in an experiment where the speed does not change.

In option 2, "mass" is independent of speed, and its numerical value is an intrinsic property of an electron, a nice advantage. The equation $E = (\text{mass})c^2$ holds, however, only for an electron at rest because the energy E varies with speed but this option's "mass" does not. [There is a whiff of inconsistency, too. The kinetic energy of the protons moving inside a nucleus does contribute measurably to the mass of a stationary nucleus (according to all options). Shouldn't the kinetic energy of a proton outside a nucleus contribute to the proton's "mass," in which case "mass" would depend on speed?]

In option 3, the right-hand side of the defining relation is a relativistic invariant, that is, its numerical value is the same in all unaccelerated reference frames. That is a nice property. The property even enables us to evaluate the expression readily when the object is an electron. We merely choose to observe the electron in its own rest frame, where its momentum is zero and its energy is m_0c^2. All factors of c cancel, and so option 3 says that the electron's inertia when the electron has zero speed is the electron's "mass" in general.

One might now think that options 2 and 3 are equivalent, but in a vital sense they are not. When applied to an electron, options 2 and 3 do indeed have identical implications: an electron's "mass" is m_0. But option 3 can also be used for "objects" that are never at rest in any frame of reference, objects to which Eq. (8) is not applicable. Most significantly, option 3 can be applied to a photon. For a photon, $E = hf$ and (momentum) $= hf/c$, and so the squares in Eq. (9) cancel each other perfectly. Thus option 3 says a photon has zero "mass." [Worth noting is that, even if one adopts option 3, the photons in a hot oven increase the oven's inertia (relative to the oven's inertia when cold). Although the photon has been assigned a zero "mass," photons do contribute positively to a system's measurable inertia. Moreover, Einstein's remark[3] of 1905, "if the theory corresponds to the facts, then radiation carries inertia between the emitting and absorbing bodies," is entirely correct; the conclusion follows from the premise and is true experimentally.]

It is not my intention to advocate any one of the three options, but I had to make a choice for this article, and it was option 1. (By specifying slow speed for the atom as observed by Bob and by taking a limit of zero speed, one can convert the derivation to option 2.) Here I offer merely a personal perspective on the options. For a physicist working regularly with four-dimensional spacetime, options 2 and 3 seem preferable. For a physicist focusing on the deep connection between inertia and energy—notions that are both historic and contemporary—option 1 seems more desirable. Of course, no conclusion about physics depends on which option is chosen; only the words used to express the conclusion may differ.

The root of the linguistic problem seems to be this: each physicist would like the short word "mass" and the unadorned symbol m for the concept that he or she uses most often. The physicist wants to avoid phrases like "the relativistic mass" [for the coefficient of $\vec{v}$ in Eq. (8)] or "the rest mass" (for m_0) or "the invariant mass" [for the right-hand side of Eq. (9)]. In speech and in writing, such phrases are cumbersome in comparison with "the mass." Until the day comes when unanimity of usage reigns, a reader will have to dig out an author's working definition and then read or translate accordingly.

If you would like to read proponents of one option or another—sometimes more passionate than reasoned—let me suggest Carl Adler,[6] Lev Okun,[7] Edwin Taylor and John Archibald Wheeler,[8] and David Bohm.[9] The differences among the options are only semantic and hence are insignificant relative to the central insight. If an atom or nucleus is at rest, all three options agree that all forms of energy contribute to the inertia of the atom or nucleus and do so in the proportion $\Delta(\text{inertia}) = \Delta E/c^2$.

If you cannot sensibly substitute the word "inertia" for the word "mass" in an author's sentence, then the author is misusing the word "mass," and the entire sentence is deeply suspect.

A Defense against Misconceptions

The central equations in this article, $\Delta E = (\Delta m)c^2$ and $E = mc^2$, have given rise to many misconceptions. How can you defend yourself and your students against them? Here is a shield.

Remember always that, in the physics of $E = mc^2$, "mass" means inertia, a reluctance to undergo a change in velocity, and that inertia is an attribute, not a thing. This provides you with a test: if you cannot sensibly substitute the word "inertia" for the word "mass" in an author's sentence, then the author is misusing the word "mass," and the entire sentence is deeply suspect.

The test will usually distinguish sense from nonsense. On certain subtle issues, however, it will not give a ringingly clear response. A case in point is the phrase "the equivalence of mass and energy." Even Einstein used this phrase (already in 1907). Let us try rewriting it: the equivalence of inertia and energy. Hmmm. What meaning can the word "equivalence" have here? The dictionary offers us several possibilities. One is "alike in significance." Can inertia (a reluctance to undergo a change in velocity) and energy (the ability to do work) be "alike in significance?" Surely not.

The dictionary also offers "equal in value." This meaning is plausible here, for the equation $E = mc^2$ says that mass and energy are proportional to each other, the proportionality

constant being c^2. Some historical digging suggests that this is what Einstein meant and what we should mean today. In 1907, Einstein published his third paper[10] on the topic of mass and energy; it carried the title, "On the inertia of energy, as implied by the principle of relativity." In a footnote[11] he remarked casually that a certain equation expresses "the principle of the equivalence of mass and energy," as though such a principle were obvious or well known. Yet this seems to be the first time the phrase appears in the scientific literature. The context suggests that Einstein meant the phrase to denote the universal numerical proportionality of inertia and energy. The concepts of inertia and energy remain distinct; their numerical values (for any given system) are proportional, with proportionality constant c^2.

For another element in the defense against misconceptions, one should bear in mind that neither inertial mass nor energy is a substance; rather, both are attributes of a physical system. "Matter," however, is most certainly a thing. An attribute cannot be converted into a thing, and so energy cannot be converted into matter. Likewise, a thing cannot be converted into an attribute, and so matter cannot be converted into energy. To be sure, these statements are nothing more than elaborations on the theme of permissible conversions discussed earlier in this article.

It is proper to say that something has energy.

It is not correct to say that something is energy.

That energy is an attribute has implications for the way we use the word. It is proper to say that something *has* energy. It is *not* correct to say that something *is* energy.

The Great Heresy

In the preceding sections, I tried hard to make only correct statements, but in teaching a class about $E = mc^2$, one may need to make a false statement and then explain why the statement is false. This tactic can expose a deeply ingrained misconception. The false statement that I have in mind, which I call *The Great Heresy*, is this:

> The equation $E = mc^2$ means that you can convert mass into energy and vice versa.

If we try the substitution test, replacing the word "mass" with the word "inertia," then our ears suggest that the claim cannot possibly be correct. The fallacy in The Great Heresy arises from a simple confusion. The heresy confuses the convertibility of matter and radiation with the parallel changes of energy and inertia. The heretic takes the word "mass" to mean "matter," as in an electron-positron pair. Moreover, the heretic takes energy to be an entity existing in its own right, rather than an attribute of some physical system, such as electromagnetic radiation. It *is* true that "you can convert matter into radiation and vice versa," but that truth is *not* what the equation $E = mc^2$ means.

As analyzed here, The Great Heresy confuses two relationships. To me, the convertibility of matter and radiation is the more surprising feature of nature. No wonder, I say to myself, that it has gotten great play. But that convertibility is distinct from the implications of $\Delta E = (\Delta m)c^2$: as the energy of a system grows, so does its inertia. And then the equation $E = mc^2$ tells us that, for any physical system, its energy E is always accompanied by an inertia whose numerical value is given by the expression E/c^2.

Another way in which The Great Heresy may arise—or may be construed—appears in the context of nuclear fission, to which we now turn.

Uranium Fission

The law of conservation of energy is all that one needs in order to understand the "energy release" in nuclear fission. A nucleus of uranium-235 absorbs a slow neutron, say, and undergoes fission. After the splitting, the electrical repulsion between the two large fragments imparts a tremendous amount of kinetic energy to the fragments as they accelerate away from each other. A few fast neutrons are emitted—that's more kinetic energy—and there is some prompt gamma radiation. Since some forms of energy have increased in value, others must have decreased. Indeed, the electrical potential energy of the fragments is less than that of the original nucleus. (The nuclear potential energy, however, actually increases as the fission reaction proceeds. The two fission fragments have more surface area than did the original uranium nucleus. Moving protons and neutrons from deep inside a nucleus to the surface layer increases their nuclear potential energy, just as moving a ball higher in the Earth's attractive gravitational field increases the ball's potential energy. The increase in nuclear potential energy is overshadowed by the large decrease in electrical potential energy.) The fission process—in which the number of protons and neutrons does not change—is primarily a transfer of energy from potential form to kinetic.

The Great Heresy may be construed as meaning that "rest mass" is converted into energy. "Rest mass," however, just means "intrinsic inertia" and that inertia is a property of the energy inherent in an object. In fission, the sum of the final rest masses is less than the sum of the initial rest masses because of some quite ordinary changes in the form in which energy is present. After fission, there is less potential energy but more kinetic energy and the energy of radiation. Because the final aggregations of protons and neutrons have less potential energy in them than did the original aggregations, they have less inertia if put at rest than did the original nucleus and neutron. So, of course, there is a reduction in the sum of the rest masses. That reduction is *not* the *source* of the energy released; rather, the reduction is just a concomitant of the release.

To be sure, the numerical value of the reduction in rest masses can be used to calculate the amount of energy released as kinetic energy and the energy of radiation. The essence of the calculation is merely $\Delta E = (\Delta m)c^2$ applied to the potential energy in the aggregations and to the inertia of those

aggregations. To learn how much the potential energy has decreased, one looks to see how much the inertia has decreased.

A Caveat about Energy

Energy, to be sure, is a subtle concept itself. The common defining statement, "energy is the ability to do work," is not the whole story. Robert Lehrman[12] emphasized that the desire to maintain a conservation law plays an essential role in how energy is defined. The present paper, however, is focused on mass and its relationship to energy, and it would be too much to explore "energy" as well. Let the incomplete phrase that I have used, "the ability to do work," suffice here, especially since it makes clear—and correctly so—that energy is an attribute, not a substance or a thing.

Looking Back

There is no uniquely right way to teach the connection between energy and inertia. There are, however, some things to avoid and at least one item to emphasize. Even in a context where "mass" means specifically "inertial mass," a semantic debate simmers over which option to chose for the technical definition of the word "mass." That controversy is entirely secondary, and one should not let it dominate the discussion. Rather, I believe, one should focus on the primary significance of the equation $\Delta E = (\Delta m)c^2$: all forms of energy contribute to an object's inertia. Or in even simpler terms, inertia is a property of energy.

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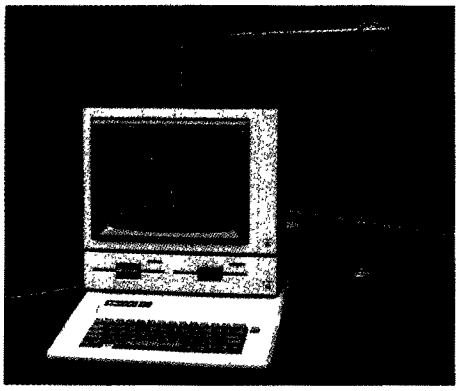

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